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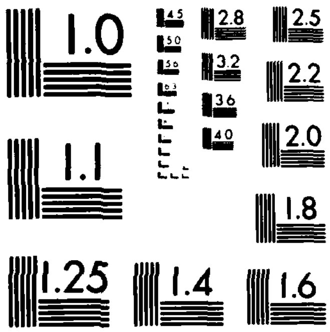

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WEAK CONVERGENCE OF A SEQUENCE OF QUEUEING AND STORAGE PROCESSES TO A SINGULAR DIFFUSION.

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1. INTRODUCTION

It has been known for a long time that heavy traffic limit theorems in queueing theory are but a special case of the so-called diffusion approximation in Physics and Genetics. Take for example Kingman's (1962) heavy traffic approximation for the stationary waiting time distribution for a sequence of GI/GI/1 queues $Q(\alpha)$ depending on a parameter α. Denote the waiting time, excluding service, of the n^{th} customer by $W(n,\alpha)$ and let $U(n,\alpha) = S(n,\alpha) - T(n,\alpha)$ where $S(n,\alpha)$ = service time of the n^{th} customer and $T(n,\alpha)$ = inter arrival time between the n^{th} and $(n+1)^{st}$ customer and assume $E(U(n,\alpha)) = -\alpha\sigma$, variance of $U(n,\alpha) = \sigma^2$, $\alpha > 0$. Then we have the following Theorem 1 (Kingman (1962)):

$$\lim_{n\to\infty} P((\alpha/\sigma)W(n,\alpha) \le x) = 1 - \exp(-2x),\ 0 \le x < \infty \text{, provided } \lim_{n\to\infty,\alpha\to 0} \alpha^2 n = \infty.$$

Somewhat later Kingman (1965) presented a more elegant but heuristic proof of this result which justifies referring to such a theorem as a diffusion approximation. It is worthwhile sketching the heuristic proof of Theorem 1 here, referring the reader to Rosenkrantz (1980) for a rigorous proof as well as an estimate of the rate of convergence. To begin with, one notes that

$$(1.1)\qquad F_{n,\alpha}(x) = P((\alpha/\sigma)W(n,\alpha) \le x) = P(\sup_{0\le t\le \alpha^2 n} y_{n,\alpha}(t) \le x)$$

where $y_{n,\alpha}(t)$ is a certain stochastic process with continuous paths. One can then show, formally at least, that

$$(1.2)\qquad \lim_{n\to\infty,\alpha\to 0} y_{n,\alpha}(t) = y(t)$$

where $y(t) = w(t) - t$. Here $w(t)$ is the standard 1-dimensional Wiener process and so $y(t)$ is the Wiener process with negative drift. It follows at once from (1.2) that

$$(1.3)\qquad \lim_{n\to\infty,\alpha\to 0} P(\sup_{0\le t\le \alpha^2 n} y_{n,\alpha}(t) \le x) = P(\sup_{0\le t<\infty} y(t) \le x)$$

and an easy calculation, see e.g. Karlin-Taylor (1975), p.361, yields the result that $P(\sup_{0\le t<\infty} y(t) \le x) = 1 - \exp(-2x)$, $0 \le x < \infty$.

Another and simpler example of a heavy traffic limit theorem is the following: let $N_n(t)$ denote the queue size of an M/M/1 queue with arrival rate λ_n, mean

service time distribution μ_n^{-1} and traffic intensity $\rho_n = \lambda_n/\mu_n$. Assume $\lambda_n = \mu_n - \delta n^{-1/2}$ for some $\delta > 0$, so $0 < \rho_n < 1$ and denote by σ_n^2 the variance of the service time distribution which in this case equals μ_n^{-2}.

THEOREM 2: Assume $\lambda = \lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \mu_n = \mu$ so $\lim \rho_n \uparrow 1$, and $\lim_{n\to\infty} \sigma_n^2 = \sigma^2$; then $\lim_{n\to\infty} N_n(nt)/\sqrt{n} = y(t)$ where $y(t)$ is the Wiener process on $R^+ = [0,\infty)$ with variance $\lambda + \sigma^2\mu^3$, negative drift δ and reflected at the origin. Theorem 2 has been extended in many ways and by many authors including Iglehart and Whitt. The survey article by Whitt (1974) is a useful reference for the reader interested in these developments.

In each of the heavy traffic limit theorems cited above the limit process has turned out to be the Wiener process with a negative drift satisfying, where appropriate, a reflecting boundary condition. Recently Yamada (1982) has given a diffusion approximation for a sequence of storage processes $X_n(t)$ where the limit process $Y(t)$ is no longer a Wiener process with a negative drift but is instead a Bessel process with negative drift. This result is of more than routine interest. It shows for example that the set of possible limit processes that can occur in queueing and storage theory is a much larger class than Theorems 1, 2 and the survey article by Whitt (1974) would lead us to believe existed. In addition Yamada's theorem (a precise version of which will be stated below as Theorem 3) offers a challenge to the traditional methods by which such limit theorems are usually proved. In particular, neither the Trotter-Kato-Kurtz method of Kurtz (1969) nor the martingale method of Papanicolaou, Stroock and Varadhan (1977) are directly applicable to this limit theorem because of some nontrivial technical problems of independent interest and the solutions of which are also of independent interest. It is the purpose of this paper to give a new and simpler proof of Yamada's theorem using some results due to Brezis, Rosenkrantz and Singer, with an appendix by P. D. Lax, (1971) which, restated in the more modern terminology of today, implies that the martingale problem for the operator corresponding to the Bessel process with drift has a unique solution - see Stroock-Varadhan (1979) and Ikeda-Watanabe (1981) for a general discussion of these ideas. It turns out however that the estimates we needed to make the martingale methods work already imply the strong convergence of the semigroups in the sense of Trotter-Kato - see Theorem 4 below. These as well as other results from Functional Analysis are collected in an appendix. We shall also use the standard notations: $C_o(R^+) = \{f: f$ bounded and continuous on $R^+ = [0,\infty)$ and $\lim_{x\to\infty} f(x) = 0\}$, $f^{(\ell)}(x) = \ell^{th}$ derivative of f, $C_o^k(R^+) = \{f \in C_o(R^+): f^{(\ell)} \in C_o(R^+),\ 1 \le \ell \le k\}$. We make $C_o(R^+)$ into a Banach space in the usual way by giving it the norm $\|f\| = \sup_{0\le x<\infty} |f(x)|$. The symbol ∎ denotes the end of a proof.

2. STATEMENT AND PROOF OF YAMADA'S DIFFUSION APPROXIMATION.

Let $X(t)$ denote the content of a dam at time t (also called a storage process) with release rate $r(x)$ and random cumulative imput $A(t)$ which is assumed to be a compound Poisson process. The jump rate λ is assumed to be finite and the cumulative distribution of the size of the jump is denoted by $F(y)$. Cinlar-Pinsky (1972) have shown that $X(t)$ may be realized as the unique solution of the stochastic integral equation

$$(2.1)\qquad X(t) = X(0) - \int_o^t r(x(x))ds + A(t), \quad \text{where}$$

$A(t) = \sum_{i=1}^{N_\lambda(t)} S_i$ where the S_i are i.i.d. with common distribution F and $N_\lambda(t)$ is a Poisson process with intensity λ. The release rate $r(x)$ is assumed to be a non-negative, non-decreasing function with domain $R^+ = [0,\infty)$, $r(0) = 0$. From now on we also assume that $\bar{r} = \lim_{x\to\infty} r(x)$ is finite. We set $\mu_i = \int_o^\infty y^i dF(y)$, $\rho = \lambda\mu_1$ and $k = \sqrt{\lambda\mu_2}$.

Following Yamada (1982) we make the following hypotheses:

$$(2.2)\qquad X_n(t) = X_n(0) - \int_o^t r_n(X_n(s))ds + A_n(t), \quad n = 1,2,\ldots$$

is a sequence of storage processes with release rates $r_n(x)$,

$A_n(t) = \sum_{i=1}^{N_{\lambda_n}(t)} S_i^n$, $P(S_i^n \le y) = F_n(y)$ satisfying the normalization conditions:

$$(2.3)\qquad \bar{r}_n \ge \rho_n, \quad \rho_n = \lambda_n\mu_1^n, \quad \mu_i^n = \int_o^\infty y^i dF_n(y)$$

$$(2.4)\qquad x(\bar{r}_n - r_n(x)) \to c < \infty, \quad \text{as } x\to\infty, \ n\to\infty$$

$$(2.5)\qquad \lim_{n\to\infty} n^{1/2}(\bar{r}_n - \rho_n)/k_n = d,$$

$$(2.6)\qquad \lim_{n\to\infty} k_n = k > 0, \quad k_n^2 = \lambda_n\mu_2^n$$

$$(2.7)\qquad \sup_{n,x\ge 0} x(\bar{r}_n - r_n(x)) = M < \infty$$

$$(2.8)\qquad X_n(0) = x_n, \qquad \lim_{n\to\infty} x_n/k_n\sqrt{n} = x\ .$$

$$(2.9)\qquad \lim_{c\to\infty} \int_{\{y>c\}} y^2 dF_n(y) = 0 \quad \text{uniformly in } n.$$

From these conditions it is easy to see that each of the following sequences is bounded: $\{\mu_2^n\}$, $\{\mu_1^n\}$, $\{\lambda_n\}$, $\{\rho_n\}$ and $\{\bar{r}_n\}$. For example (2.9) implies that $\{\mu_2^n\}$ is a bounded sequence and *a fortiori* so is $\{\mu_i^n\}$. This together with (2.6) implies $\{\lambda_n\}$ is bounded and the other statements are proved in a similar fashion.

THEOREM 3 (Yamada): Set $Y_n(t) = X_n(nt)/k_n\sqrt{n}$ and assume conditions (2.3) through (2.9) hold and that $\lim_{n\to\infty} Y_n(0) = x$. Then $Y_n(t)$ converges weakly to a Bessel process with negative drift $Y(t)$, starting at x. $Y(t)$ is a (Markov) diffusion process on $R^+ = [0,\infty)$ whose infinitesimal generator is given by

$$Gf(x) = (1/2)f''(x) + (c/k^2)(f'(x)/x) - df'(x). \tag{2.10}$$

Remarks: This is not the form in which Yamada states his theorem. Specifically, he shows that $Y(t) = \sqrt{Z(t)}$ where $Z(t)$ is the unique solution to the stochastic integral equation:

$$Z(t) = Z(0) + \int_0^t (K - 2d\sqrt{Z(s)})ds + 2\int_0^t \sqrt{Z(s)}dw(s) \tag{2.11}$$

where $K = 1 + 2c/k^2$ and w is the standard Wiener process. Thus $Z(t)$ satisfies the stochastic differential equation

$$\begin{cases} dZ(t) = (K - 2d\sqrt{Z(t)})dt + 2\sqrt{Z(t)}dw(t) \\ \qquad\;\; = b(Z(t))dt + a(Z(t))dw(t) \quad \text{with} \\ b(x) = (K - 2d\sqrt{x}), \quad x \geq 0 \quad \text{and} \quad a(x) = 2\sqrt{x} \end{cases} \tag{2.12}$$

Notice that neither $a(x)$ nor $b(x)$ (when $d \neq 0$) are Lipschitz continuous and so the existence of a unique solution to the stochastic differential equation (2.12) is not a trivial matter. The existence of a unique solution is however a consequence of a more general result due to Okabe and Shimizu (1975). Before proceeding to our own proof let us sketch the idea behind Yamada's proof. He first shows that the processes $Y_n(t)$ are tight in $D[0,T]$ and that if $Y(t)$ is any limit then $Z(t) = Y(t)^2$ solves the martingale problem:

$$f(Z(t)) - f(Z(0)) - \int_0^t (K - 2d)\sqrt{Z(s)})f'(Z(s))ds - 2\int_0^t \sqrt{Z(s)}f''(Z(s))ds \quad \text{is a zero mean} \tag{2.13}$$

martingale for every $f \in C_K^2(R)$. $C_K^2(R)$ is the set of twice continuously differentiable functions, with compact support. This shows that every weak limit solves the martingale problem (2.13) which, thanks to the results of Okabe-Shimuzu, op. cit, is known to have a unique solution. The proof that $Z(t)$ is a solution to the martingale problem (2.13) is almost 5 pages long and the proof that the processes $\{Y_n(t)\}$ form a tight sequence is nearly 6 pages long. It is the purpose of this paper to give an alternative proof of this result which we believe to be easier to follow and is also somewhat shorter. First we shall give a heuristic proof and put in the (tedious) details elsewhere.

We begin by observing that $Y_n(t)$ is for each n a Markov process on the half line $R^+ = [0,\infty)$ with infinitesimal generator G_n given by

$$
(2.14)\quad
\begin{cases}
G_n f(x) = -(\sqrt{n}/k_n)\, r_n(k_n\sqrt{n}\cdot x) f'(x) + n\lambda_n \int_0^\infty [f(x+y) - f(x)]\, dH_n(y) \\
\qquad \text{for } x > 0 \quad \text{and} \\
G_n f(0) = n\lambda_n \int_0^\infty [f(y) - f(0)]\, dH_n(y). \\
\text{Here } H_n(y) = F_n(k_n\sqrt{n}\cdot y).
\end{cases}
$$

See for example Cinlar-Pinsky (1972), Harrison-Resnick (1976) or Rosenkrantz (1981) where the operators G_n and their domains (both strong and weak) are discussed in some detail.

DEFINITION: $D(G) = \{f \in C_0^2(R^+) : f'(0) = 0\}$, where the operator G is defined at (2.10).

Later on, in Appendix A, we will show that $D(G)$ is the domain of the strong infinitesimal generator of the semi group $T(t)f(x) = E_x(f(Y(t)))$. Of course, characterizing $D(G)$ is not, in general, an easy matter but in the special case $d = 0$ this was already done by Brezis et al. (1971). The extension of their results to the case $d \neq 0$ is carried out in this paper by showing that the operator $Cf(x) = -df'(x)$ is <u>relatively bounded</u> with respect to the Bessel operator

$$
(2.15)\qquad Bf(x) = (1/2)f''(x) + (\gamma/x)f'(x), \quad \gamma > -1/2,
$$

in the sense of Kato (1976) cf. Appendix A. With these preliminaries out of the way we can give a quick heuristic proof of Yamada's theorem by deriving the

LEMMA 1: For every $f \in D(G)$ and $x > 0$ we have $\lim_{n\to\infty} G_n f(x) = Gf(x)$; the convergence is uniform on every interval of the form $[\delta,\infty)$, $\delta > 0$ and $\sup_n \|G_n f(x)\| < \infty$.

PROOF: Using the Taylor expansion $f(x+y) - f(x) = f'(x)y + (1/2)f''(x)y^2 + R(x,y)$ where $R(x,y) = (1/2)(f''(\xi(y)) - f''(x))$ and $x \le \xi(y) \le x + y$, we see that

$$
n\lambda_n \int_0^\infty [f(x + y) - f(x)]\, dH_n(y) =
$$

$$
n\lambda_n f'(x) \int_0^\infty y\, dH_n(y) + (1/2)n\lambda_n f''(x) \int_0^\infty y^2\, dH_n(y) + R(n)
$$

where $|2R(n)| \le n\lambda_n \int_0^\infty |[f''(\xi(y)) - f''(x)]| y^2\, dH_n(y)$. In a moment we will show that $\lim_{n\to\infty} R(n) = 0$. On the other hand $\int_0^\infty y\, dH_n(y) = \mu_1^n/k_n\sqrt{n}$ and $\int_0^\infty y^2\, dH_n(y) = \mu_2^n/k_n^2 n$ so

$$
(2.16)\qquad G_n(f(x)) = [-(\sqrt{n}/k_n)r_n(k_n\sqrt{n}\cdot x) + (\sqrt{n}\mu_n/k_n)]f'(x) + (1/2)f''(x) + R(n),
$$

since $n\lambda_n\mu_1^n/k_n\sqrt{n} = \sqrt{n}\mu_n/k_n$ and $n\lambda_n\mu_2^n/k_n^2 n = 1$ - see (2.3) and (2.6). Adding and

subtracting the term $(\sqrt{n}/k_n)\bar{r}_n f'(x)$ to the right hand side of (2.16) we obtain

$$G_n f(x) = (\sqrt{n}/k_n)(\bar{r}_n - r_n(k_n\sqrt{n}\cdot x))f'(x) + (\sqrt{n}/k_n)(\rho_n - \bar{r}_n)f'(x)$$

$$+ (1/2)f''(x) + R(n).$$

For $x > 0$ we have $(\sqrt{n}/k_n)(\bar{r}_n - r_n(k_n\sqrt{n}\cdot x))f'(x) = (k_n\sqrt{n}\cdot x/k_n^2)(\bar{r}_n - r_n(k_n\sqrt{n}\cdot x))f'(x)/x$ consequently (2.4)(2.6),(2.7) imply that for $x > 0$ $\lim_{n\to\infty} (\sqrt{n}/k_n)(r_n - r_n(k_n\sqrt{n}\cdot x))f'(x) = (c/k^2)f'(x)/x$ and the convergence is uniform on the interval $[\delta,\infty)$. Hypothesis (2.7) implies that the term is uniformly bounded in n and x. Similarly condition (2.5) implies $\lim_{n\to\infty} (\sqrt{n}/k_n)(r_n - \bar{\rho}_n)f'(x) = -df'(x)$. Thus the lemma will be proved if we can show that $\lim_{n\to\infty} R(n) = 0$, where

$$|2R(n)| \le n\lambda_n\int_0^\varepsilon |f''(\xi(y)) - f''(x)|y^2 dH_n(y) + n\lambda_n\int_\varepsilon^\infty |f''(\xi(y)) - f''(x)|y^2 dH_n(y).$$

Now for ε small enough $|f''(\xi(y)) - f''(x)| < \delta$ and this together with the fact that $n\lambda_n \int_0^\varepsilon y^2 dH_n(y) \le n\lambda_n \int_0^\infty y^2 dH_n(y) = 1$ implies that the first summand in the expression above can be made arbitrarily small. As for the second summand a change of variable yields the formula $n\lambda_n\int_\varepsilon^\infty y^2 dH_n(y) = (\lambda_n/k_n^2)\int_{k_n\sqrt{n}\cdot\varepsilon}^\infty z^2 dF_n(z)$ which goes to zero by hypothesis (2.9) and the fact that both λ_n and k_n^2 are bounded. $\Box$

It is easy to see that $\lim_{n\to\infty} G_n f(0) \ne Gf(0)$. Because $Gf(0) = (1/2)f''(0) + (c/k^2)f''(0) - df'(0) = (1/2 + c/k^2)f''(0)$ since $f'(0) = 0$ and $f \in C_0^2(R^+)$ implies $f''(0) = \lim_{x\to 0} \frac{f'(x)}{x}$. On the other hand (by (2.14)) $G_n f(0) = n\lambda_n\int_0^\infty (f(y) - f(0))dH_n(y)$ and using a two term Taylor expansion as before we get that $\lim_{n\to\infty} G_n f(0) = (1/2)f''(0)$. Thus the only time $G_n f(x)$ converges $Gf(x)$ for all $x \in R^+$ is in the special case $c = 0$. i.e. when the limiting process $Y(t)$ is the Wiener process with a negative drift reflected at the origin. This phenomenon of convergence of the generators except at certain exceptional points is quite common and occurs even in the example of Theorem 2 - cf. Burman (1979) p.17. Nevertheless, it has been observed by several authors including Papanicolaou, Stroock, Varadhan (1975), Burman (1979) that weak convergence of $Y_n(t)$ to $Y(t)$ can be proved, provided one can show that the occupation time of the exceptional set by the process $Y_n(t)$ can be made arbitrarily small as $n \to \infty$. In the present context we must estimate $\int_0^T I_{[0,\delta]}(Y_n(s))ds$ which is the occupation time of the set $[0,\delta]$ by the process $Y_n(t)$.

LEMMA 2: Under the hypotheses of Theorem 3 there exists for any $\epsilon > 0$ a $\delta > 0$ such that

$$(2.17) \qquad \limsup_{n\to\infty} E_x\left(\int_0^T I_{[0,\delta]}(Y_n(s))ds \right) \leq \epsilon .$$

Setting aside the proof of (2.17) for the moment let us show that this implies strong convergence of the semi groups.

THEOREM 4: Under the hypotheses of Theorem 3

$$(2.18) \qquad \lim_{n\to\infty} \| E_x(f(Y_n(t)) - E_x f(Y(t)) \| = \lim_{n\to\infty} \| T_n(t)f(x) - T(t)f(x) \| = 0,$$

where the convergence is uniform for $t \in$ compact subsets of R^+. Before proceeding to the proof of Theorem 4 we need a result due to the author, Rosenkrantz (1981), characterizing the domains $D(G_n)$ of the integro-differential operators G_n defined at (2.14).

LEMMA 3: Let G_n denote the strong infinitesimal generator of the normalized storage processes. Then

Case 1: If $r_n(x) = \bar{r}_n$, $x > 0$, $r_n(0) = 0$ we have

$$(2.19) \qquad D(G_n) = \{f \in C_0^1(R^+) : f'(0) = 0\}$$

$$(2.20) \qquad \text{Case 2:} \quad D(G_n) = \{f \in C_0(R^+) : r_n(x)f'(x) \in C_0(R^+),\ \lim_{x\to 0} r_n(x)f'(x) = 0\} .$$

PROOF: This theorem is proved in exactly the same way as Theorem 4.6 on p. 219 of Rosenkrantz (1981). $\square$

Clearly $D(G) \subset D(G_n)$ and hence for every $f \in D(G)$ we have the representation

$$(2.21) \qquad T_n(t)f(x) - T(t)f(x) = \int_0^t T_n(t-s)(G_n - G)T(s)f(x)ds,$$

cf. Burman (1979) p. 14, formula 2.2.

We pause to introduce some notation: If $g(x)$ is a function set $g_\delta(x) = g(x)$ if $x \in [\delta,\infty)$ and 0 otherwise and put $\tilde{g}_\delta(x) = g(x) - g_\delta(x)$; so $g_\delta(x) + \tilde{g}_\delta(x) = g(x)$. Thus $(G_n - G)T(s)f(x) = [(G_n - G)T(s)f]_\delta(x) + \widetilde{[(G_n - G)T(s)f]}_\delta(x)$ and therefore

$$\|T_n(t)f(x) - T(t)f(x)\| \leq \int_0^t \|[(G_n - G)T(s)f]_\delta(x)ds + \|\int_0^t T_n(t-s)\widetilde{[(G_n - G)T(s)f]}_\delta(x)ds\|$$

since $T_n(t)$ is a contraction semi group.

For $f \in D(G)$ the *apriori* estimate (A.8) and Lemma 1 together imply $\lim_{n\to\infty} (G_n - G)T(s)f(x) = 0$ uniformly on $[\delta,\infty)$ and uniformly in s, $0 \le s \le t$. Consequently $\lim_{n\to\infty} \int_0^t \|[(G_n - G)T(s)f]_\delta(x)\| ds = 0$. Similarly, $[(G_n - G)T(s)f]_\delta(x) \neq 0$ only on the set $[0,\delta]$ and since by Lemma 1 and (A.8) $\|G_nT(s)f\|$ and $\|GT(s)f\|$ are both uniformly bounded we conclude $\left|\int_0^t T_n(t - s)[(G_n - G)T(s)f]_\delta(x)ds\right| = \left|E_x\int_0^t [(G_n - G)T(s)f]_\delta(Y_n(t - s))ds\right| \le c'E_x\left(\int_0^t I_{[0,\delta]}(Y_n(s))ds\right)$ where $c' = \sup_{n, 0\le s\le t} \{\|G_nT(s)f(x)\| + \|GT(s)f(x)\|\}$. We now apply Lemma 2 and choose δ so small that $\limsup_{n\to\infty} E_x\left(\int_0^t I_{[0,\delta]}(Y_n(s))ds\right) \le \varepsilon \cdot c^{-1}$ from which it follows at once that $\limsup_{n\to\infty} \|T_n(t)f(x) - T(t)f(x)\| \le \varepsilon$ uniformly for $t \in$ compact subsets of R^+. □

We now turn to the proof of Lemma 3. Following Yamada let $\bar{Y}_n(t)$ denote the storage process with $\bar{r}_n(x) = \bar{r}_n$, $x > 0$ and $\bar{r}_n(0) = 0$. Since $\bar{r}_n(x) \ge r_n(x)$ it is clear that $\bar{Y}_n(t) \ge Y(t)$ and in particular

$$E_x\left(\int_0^t I_{[0,\delta]}(Y_n(s))ds\right) \le E_x\left(\int_0^t I_{[0,\delta]}(\bar{Y}_n(s))ds\right) .$$

Thus to prove Lemma 3 it suffices to prove that

$$\limsup_{n\to\infty} E_x\left(\int_0^t I_{[0,\delta]}(\bar{Y}_n(s))ds\right) \le \varepsilon . \tag{2.22}$$

It is convenient to split the proof into two parts:

$$\lim_{n\to\infty} E_x\left(\int_0^t I_{[0]}(\bar{Y}_n(s))ds\right) = 0 \tag{2.23}$$

$$\limsup_{n\to\infty} E_x\left(\int_0^t I_{(0,\delta]}(\bar{Y}_n(s))ds\right) \le \varepsilon . \tag{2.24}$$

PROOF OF (2.23): The infinitesimal generator G_n' of $\bar{Y}_n(t)$ is

$$G_n'f(x) = -(\sqrt{n}/k_n)\bar{r}_n f'(x) + n\lambda_n \int_0^\infty [f(x + y) - f(x)]dH_n(y), \quad x > 0$$

$$G_n' f(0) = n\lambda_n\int_0^\infty [f(y) - f(0)]dH_n(y).$$

Applying Dynkin's formula as in Theorem 3.1 p. 216 of Rosenkrantz (1981), leads to the formula

$$E_x(\bar{Y}_n(t)) = x - (\sqrt{n}/k_n)(\bar{r}_n - \rho_n)t + (\sqrt{n}/k_n)\bar{r}_n\, E_x\left(\int_0^t I_{[0]}(Y_n(s))ds\right) . \tag{2.25}$$

In the appendix it will be shown that $\sup_{0\le s\le t} E_x(\bar{Y}_n(s)) < \infty$ for every $t > 0$ and hence

(2.26) $$(\sqrt{n}/k_n)\bar{r}_n E_x\left(\int_0^t I_{[0]}(Y_n(s))ds\right) = E_x(\bar{Y}_n(t)) - x + (\sqrt{n}/k_n)(\bar{r}_n - \rho_n)t.$$

By (2.7) $\lim_{n\to\infty} (\sqrt{n}/k_n)(\bar{r}_n - \rho_n)t = dt$ so the right hand side of (2.26) is bounded whilst $\lim_{n\to\infty} (\sqrt{n}/k_n)\bar{r}_n = +\infty$, consequently

(2.27) $$E_x\left(\int_0^t I_{[0]}(Y_n(s))ds\right) = O(n^{-1/2}).$$

Turning now to the proof of (2.24) we must consider separately the case when $d > 0$ and when $d = 0$.

Case 1: $d = \lim_{n\to\infty} (\sqrt{n}/k_n)(\bar{r}_n - \rho_n) > 0$.

LEMMA 4: For every $\alpha > 0$ the function $f_\alpha(x) = [1 - (x/\alpha)]^+$ is in the domain of the weak infinitesimal generator $\tilde{G}_n'$.

PROOF: See Harrison-Resnick (1976). Of course $\tilde{G}_n'$ is an extension of G_n' and $f_\alpha(x)$ is Lipschitz continuous with $|f_\alpha(x + y) - f_\alpha(x)| \le y \cdot \alpha^{-1}$, $f_\alpha'(x) = -\alpha^{-1}$ on $[0,\alpha]$. Thus,

(2.28) $$\tilde{G}_n' f_\alpha(x) = (\sqrt{n}/k_n)\bar{r}_n \alpha^{-1} + n\lambda_n \int_0^\infty [f_\alpha(x + y) - f_\alpha(x)]dH_n(y), \quad 0 < x \le \alpha,$$

(2.29) $|\tilde{G}_n' f_\alpha(0)| \le (\sqrt{n}/k_n)\lambda_n \cdot \alpha^{-1}$, $\tilde{G}_n' f_\alpha(x) = 0$, $x > \alpha$. In particular

$$\tilde{G}_n' f_\alpha(x) \ge (\sqrt{n}/k_n)\bar{r}_n \cdot \alpha^{-1} - n\lambda_n \int_0^\infty \alpha^{-1} \cdot y dH_n(y)$$
$$= \alpha^{-1}(\sqrt{n}/k_n)(\bar{r}_n - \rho_n) \quad \text{on} \quad (0,\alpha].$$

Now for large n, $(\sqrt{n}/k_n)(\bar{r}_n - \rho_n) \ge d/2 > 0$ and this implies $\tilde{G}_n' f_\alpha(x) \ge d/2\alpha$ on $(0,\alpha]$ provided n is large enough. Notice that $\|f_\alpha(x)\| \le 1$ and hence $T_n(t)f_\alpha(x) - f_\alpha(x) = \int_0^t T_n(s)\tilde{G}_n' f_\alpha(x)ds$ implies $\left|E_x\left(\int_0^t \tilde{G}_n' f_\alpha(\bar{Y}_n(s))ds\right)\right| \le 2$. On the other hand $E_x\left(\int_0^t \tilde{G}_n' f_\alpha(\bar{Y}_n(s))ds\right) = E_x\left(\int_0^t \tilde{G}_n' f_\alpha(\bar{Y}_n(s)) I_{[0]}(\bar{Y}_n(s))ds\right)$ $+ E_x\left(\int_0^t \tilde{G}_n' f_\alpha(\bar{Y}_n(s)) I_{(0,\alpha]}(\bar{Y}_n(s))ds\right)$. From (2.27) and (2.29) we see at once that $\left|E_x\left(\int_0^t \tilde{G}_n' f_\alpha(\bar{Y}_n(s)) I_{[0]}(\bar{Y}_n(s))ds\right)\right|$ is bounded, by M say, as $n \to \infty$. On the interval $(0,\alpha]$ however, $\tilde{G}_n' f_\alpha(x) \ge (d/2\alpha)$ and therefore $E_x\left(\int_0^t \tilde{G}_n' f_\alpha(\bar{Y}_n(s)) I_{(0,\alpha]}(\bar{Y}_n(s))ds\right)$ $\ge (d/2\alpha)E_x\left(\int_0^t I_{(0,\alpha]}(\bar{Y}_n(s))ds\right)$. Therefore as $n \to \infty$ we get

$$\limsup_{n\to\infty} E_x\left(\int_0^t I_{(0,\alpha]}(\bar{Y}_n(s))ds\right) \le (2 + M)2\alpha/d .$$

The proof is now completed by choosing $\delta \le \epsilon d/(4 + 2M)$.

Case 2: $d = 0$. In this case $\lim_{n\to\infty} G_n' f(x) = (1/2)f''(x)$ for every $f \in D(G) = \{f \in C_0^2(R^+): f'(0) = 0\}$ i.e. the limit process in this case is reflecting Brownian motion $|w(t)|$. Thus the original Trotter-Kato theorem itself implies that $\lim_{n\to\infty} \| E_x(f(\bar{Y}_n(t)) - E_x f(|w(t)|)\| = 0$. It is a consequence of a theorem of Aldous (1978) that $\bar{Y}_n(t)$ converges weakly to $|w(t)|$ or if one prefers, the weak convergence may be deduced from a more general result due to Kurtz (1981), Theorem 4.4. It is well known that reflecting Brownian motion has a local time $\ell(t,y,\omega)$ and therefore $\int_0^t I_{[0,\delta]}(|w(s)|)ds = \int_0^\delta \ell(t,y,\cdot)dy \le t$, where $\ell(t,y,\omega)$ is jointly continuous in (t,y) for each ω. By Lebesgue's dominated convergence theorem then we have $\lim_{\delta\to 0} E\left\{\int_0^\delta \ell(t,y,\cdot)dy\right\} = 0$ and so given any $\epsilon > 0$ $\exists\delta > 0$ such that $E\left\{\int_0^t I_{[0,\delta]}(|w(s)|)ds\right\} < \epsilon$.

Let us denote by P_n and P the measures induced on $D[0,T]$ by the $\bar{Y}_n(t)$ and $|w(t)|$ processes respectively. It is well known that the functional $\omega \to \int_0^t I_{[0,\delta]}(\omega(s))ds$, where ω is a path in $D[0,T]$, is continuous almost everywhere with respect to the measure P, cf. Billingsley (1968), pp. 230-231. This fact together with the weak convergence of P_n to P and Theorem (5.2iii) p. 31 of Billingsley, op. cit., imply

$$\text{(2.30)} \qquad \lim_{n\to\infty} E_x\left\{\int_0^t I_{[0,\delta]}(\bar{Y}_n(s))ds\right\} = E_x\left\{\int I_{[0,\delta]}(|w(s)|)ds\right\} < \epsilon. \qquad \square$$

The proof of Theorem 4 is now complete.

APPENDIX

Let $Bf(x) = \ldots$
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(1971) that B act...
tinuous, contracti...
timate was also obt...
a more general resu...

LEMMA: For every ...
such that

(A.1) $\|f''\| \le$

We next observe tha...
fined by (2.15) and
clearly $D(C) \supset D(\ldots$

THEOREM: There exi...
the inequality

(A.2) $\|Cf\| \le$

REMARK: When (A.2...
respect to B - see

PROOF: Let $\|g\|_{[a,\ldots}$
$\sup_k \|g\|_{[k,k+1]}$ where
The proof of in...
(1.13).

(A.3) $\|f'\|_{[a,b}$

Specializing (A.3) t...

(A.4) $\|f'\|_{[k,k\ldots}$

If now $f \in C_0^2(\ldots$
hence

(A.5) $\|f'\|_{[k,k\ldots}$

APPENDIX

Let $Bf(x) = (1/2)f''(x) + (\gamma/x)f'(x)$, $\gamma > -(1/2)$ denote the Bessel operator acting on the domain $D(B) = \{f \in C_0^2(R^+) : f'(0) = 0\}$. It was shown in Brezis, et al. (1971) that B acting on $D(B)$ generates a positivity preserving, strongly continuous, contraction semi group $T_1(t) : C_0(R^+) \to C_0(R^+)$. The following *a priori* estimate was also obtained (see Theorem (A.1) p. 411 of Brezis et al. (1971), where a more general result is given):

LEMMA: For every $f \in D(B)$ there exists a constant $\beta > 0$, depending only on γ, such that

(A.1) $\|f''\| \le \beta\|Bf\|$.

We next observe that the operator $Gf = Bf + Cf$ where B is Bessel operator defined by (2.15) and $Cf = -df'$, i.e., G is a perturbation of the operator B; clearly $D(C) \supset D(B)$.

THEOREM: There exist constants $a > 0$, $0 < b < 1/2$ such that for every $f \in D(B)$ the inequality

(A.2) $\|Cf\| \le a\|f\| + b\|Bf\|$, holds.

REMARK: When (A.2) holds the operator C is said to be *relatively bounded with respect to* B - see Kato (1976), p. 190.

PROOF: Let $\|g\|_{[a,b]} = \sup_{a \le x \le b} |g(x)|$ and observe that, for $g \in C_0(R^+)$, $\|g\| = \sup_k \|g\|_{[k,k+1]}$ where the sup is taken over all non-negative integers $k = 0,1,2,\ldots$. The proof of inequality (A.3) below is to be found in Kato, op. cit. p. 192, formula (1.13).

(A.3) $\|f'\|_{[a,b]} \le [(b-a)/(n+2)]\cdot\|f''\|_{[a,b]} + [2(n+1)/(b-a)]\cdot\|f\|_{[a,b]}$

for every $f \in C^2[a,b]$ and every $n \ge 1$.

Specializing (A.3) to the special case $[a,b] = [k,k+1]$ yields

(A.4) $\|f'\|_{[k,k+1]} \le (n+2)^{-1}\|f''\|_{[k,k+1]} + 2(n+1)\|f\|_{[k,k+1]}$.

If now $f \in C_0^2(R^+)$ we have $\|f''\|_{[k,k+1]} \le \|f''\|$ and $\|f\|_{[k,k+1]} \le \|f\|$ and hence

(A.5) $\|f'\|_{[k,k+1]} \le (n+2)^{-1}\|f''\| + 2(n+1)\|f\|$.

Consequently for every $f \in D(B)$ we have

$$\|f''\| = \sup_k \|f''\|_{[k,k+1]} \le (n+2)^{-1}\|f''\| + 2(n+1)\|f\| \quad \text{and in particular}$$

(A.6) $$\|Cf\| \le d(n+2)^{-1}\|f''\| + 2d(n+1)\|f\| \le \beta d(n+2)^{-1}\|Bf\| + 2d(n+1)\|f\|$$

where we used (A.1) in the last step.

Thus by choosing $n > 2\beta d - 2$ we have $b = \beta d(n+2)^{-1} < \frac{1}{2}$ and this completes the proof (A.2) with $a = 2d(n+1)$. □

The following *a priori* estimate is also an easy consequence of the above calculation:

(A.7) $$\|f''\| \le 2\beta\|Gf\| + 4\beta d(n+1)\|f\|.$$

PROOF: Since $Bf = Gf - Cf$ we have from (A.1) and (A.6) that $\|f''\| \le \beta\|Gf\| + \beta\|Cf\| \le \beta\|Gf\| + \beta d(n+2)^{-1}\|f''\| + 2d\beta(n+1)\|f\|$. Since $\beta d(n+2)^{-1} < \frac{1}{2}$ we have $(1 - \beta d(n+2)^{-1})\|f''\| \le \beta\|Gf\| + 2\beta d(n+1)\|f\|$ and hence $\|f''\| \le 2\beta\|Gf\| + 4\beta d(n+1)\|f\|$. □

Combining all these estimates together with Theorem 2.7 of Kato p. 501 we arrive at the

THEOREM: The operator $G = B + C$ generates a positivity preserving, strongly continuous contraction semi group $T(t): C_0(R^+) \to C_0(R^+)$ with domain $D(G) = D(B) = \{f \in C_0^2(R^+): f'(0) = 0\}$. Moreover for every $f \in D(G)$ we have the following *a priori* estimate: $\|f''\| \le 2\beta\|Gf\| + 4\beta d(n+1)\|f\|$. In particular if $f \in D(G)$ then $T(s)f \in D(G)$ and therefore

(A.8) $$\begin{aligned}\|(\partial^2/\partial x^2)T(s)f(x)\| &\le 2\beta\|GT(s)f\| + 4\beta d(n+1)\|f\| \\ &\le 2\beta\|T(s)Gf\| + 4\beta d(n+1)\|f\| \\ &\le 2\beta\|Gf\| + 4\beta d(n+1)\|f\|.\end{aligned}$$

We have used the facts that $T(s)$ commutes with its infinitesimal generator G and that $T(s)$ is a contraction. Notice that the right hand is independent of s.

We next turn our attention to deriving the estimate:

(A.9) $$\sup_{0\le s\le t} E_x(\bar{Y}_n(s)^2) \le x^2 + t.$$

This clearly implies $\sup_{0\le s\le t} E_x(\bar{Y}_n(s)) < \infty$ which is all we needed to derive (2.27).

PROOF OF (A.9): Let $U(t,x) = x^2 - t$ and observe that

$$G_n'U(t,x) = -2(\sqrt{n}/k_n)\tilde{r}_n x + n\lambda_n \int_0^\infty (2xy + y^2)\,dH_n(y)$$

$$= 1 - (2\sqrt{n}/k_n)x(\bar{r}_n - \rho_n) \leq 1 \quad \text{on} \quad R^+ .$$

Thus $[(\partial U/\partial t) + G_n']U(t,x) = -1 + G_n'U(t,x) \leq 0$; consequently $\bar{Y}_n(t) - t^2$ is a supermartingale. Thus $E_x(\bar{Y}_n(t)^2 - t) \leq x^2$ or $E_x(\bar{Y}_n(t)^2) \leq x^2 + t$. $\square$

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

It has been known for a long time that heavy traffic limit theorems in queueing theory are but a special case of the so-called diffusion approximation in Physics and Genetics. Take for example Kingman's (1962) heavy traffic approximation for the stationary waiting time distribution for a sequence of GI/GI/1 queues $Q(\alpha)$ depending on a parameter α. Denote the waiting time, excluding service, of the n^{th} customer by $W(n,\alpha)$ and let $U(n,\alpha) = S(n,\alpha) - T(n,\alpha)$ where $S(n,\alpha)$ = service time of the n^{th} customer and $T(n,\alpha)$ = inter arrival time between the n^{th} and $(n+1)^{st}$ customer and assume $E(U(n,\alpha)) = -\alpha\sigma$, variance of $U(n,\alpha) = \sigma^2$, $\alpha > 0$. Then we have the following Theorem 1 (Kingman (1962)):

$$\lim_{n\to\infty} P((\alpha/\sigma)W(n,\alpha) \le x) = 1 - \exp(-2x),\quad 0 \le x < \infty, \text{ provided } \lim_{n\to\infty,\ \alpha\to 0} \alpha^2 n = \infty.$$

Somewhat later Kingman (1965) presented a more elegant but heuristic proof of this result which justifies referring to such a theorem as a diffusion approximation. It is worthwhile sketching the heuristic proof of Theorem 1 here, referring (CONTINUED)

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ITEM #19, ABSTRACT, CONTINUED:

the reader to Rosenkrantz (1980) for a rigorous proof as well as an estimate of the rate of convergence. To begin with, one notes that (1.1) $F_{n,\alpha}(x) = P((\alpha/\sigma)W(n,\alpha) \le x) = P(\sup_{0<t<\alpha^2 n} y_{n,\alpha}(t) \le x)$ where $y_{n,\alpha}(t)$ is a certain stochastic process with continuous paths. One can then show, formally at least, that (1.2) $\lim_{n\to\infty,\alpha\to 0} y_{n,\alpha}(t) = y(t)$ where $y(t) = w(t) - t$. Here $w(t)$ is the standard 1-dimensional Wiener process and so $y(t)$ is the Wiener process with negative drift. It follows at once from (1.2) that (1.3) $\lim_{n\to\infty,\alpha\to 0} P(\sup_{0\le t\le\alpha^2 n} y_{n,\alpha}(t) \le x) = P(\sup_{0<t<\infty} y(t) \le x)$ and an easy calculation, see e.g. Karlin-Taylor (1975), p.361, yields the result that $P(\sup_{0<t<\infty} y(t) \le x) = 1 - \exp(-2x)$, $0 \le x < \infty$. Another and simpler example of a heavy traffic limit theorem is the following: let $N_n(t)$ denote the queue size of an M/M/1 queue with arrival rate λ_n, mean service time distribution μ_n^{-1} and traffic intensity $\rho_n = \lambda_n/\mu_n$. Assume $\lambda_n = \mu_n - \delta n^{-1/2}$ for some $\delta > 0$, so $0 < \rho_n < 1$ and denote by σ_n^2 the variance of the service time distribution which in this case equals μ_n^{-2}.

THEOREM 2: Assume $\lambda = \lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \mu_n = \mu$ so $\lim \rho_n \uparrow 1$, and $\lim_{n\to\infty} \sigma_n^2 = \sigma^2$; then $\lim_{n\to\infty} N_n(nt)/\sqrt{n} = y(t)$ where $y(t)$ is the Wiener process on $R^+ = [0,\infty)$ with variance $\lambda + \sigma^2\mu^3$, negative drift δ and reflected at the origin. Theorem 2 has been extended in many ways and by many authors including Iglehart and Whitt. The survey article by Whitt (1974) is a useful reference for the reader interested in these developments.

In each of the heavy traffic limit theorems cited above the limit process has turned out to be the Wiener process with a negative drift satisfying, where appropriate, a reflecting boundary condition. Recently Yamada (1982) has given a diffusion approximation for a sequence of storage processes $X_n(t)$ where the limit process $Y(t)$ is no longer a Wiener process with a negative drift but is instead a Bessel process with negative drift. This result is of more than routine interest. It shows for example that the set of possible limit processes that can occur in queueing and storage theory is a much larger class than Theorems 1, 2 and the survey article by Whitt (1974) would lead us to believe existed. In addition Yamada's theorem (a precise version of which will be stated below as Theorem 3) offers a challenge to the traditional methods by which such limit theorems are usually proved. In particular, neither the Trotter-Kato-Kurtz method cf Kurtz (1969) nor the martingale method of Papnicolaou, Stroock and Varadhan (1977) are directly applicable to this limit theorem because of some nontrivial technical problems of independent interest and the solutions of which are also of independent interest. It is the purpose of this paper to give a new and simpler proof of Yamada's theorem using some results due to Brezis, Rosenkrantz and Singer, with an appendix by P. D. Lax, (1971) which, restated in the more modern terminology of today, implies that the martingale problem for the operator corresponding to the Bessel process with drift has a unique solution - see Stroock-Varadhan (1979) and Ikeda-Watanabe (1981) for a general discussion of these ideas. It turns out however that the estimates we needed to make the martingale methods work already imply the strong convergence of the semigroups in the sense of Trotter-Kato - see Theorem 4 below. These as well as other results from Functional Analysis are collected in an appendix. We shall also use the standard notations: $C_o(R^+) = \{f: f$ bounded and continuous on $R^+ = [0,\infty)$ and $\lim_{x\to\infty} f(x) = 0\}$, $f^{(\ell)}(x) = \ell^{th}$ derivative of f, $C_o^k(R^+) = \{f \in C_o(R^+): f^{(\ell)} \in C_o(R^+), 1 \le \ell \le k\}$. We make $C_o(R^+)$ into a Banach space in the usual way by giving it the norm $\|f\| = \sup_{0\le x<\infty} |f(x)|$. The symbol ∎ denotes the end of a proof.

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